

# $B$ Decays in the Upsilon Expansion

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## Abstract

Theoretical predictions for  $B$  decay rates are rewritten in terms of the Upsilon meson mass instead of the  $b$  quark mass, using a modified perturbation expansion. The theoretical consistency is shown both at low and high orders. This method improves the behavior of the perturbation series for inclusive and exclusive decay rates, and the largest theoretical error in the predictions coming from the uncertainty in the quark mass is eliminated. Applications to the determination of CKM matrix elements, moments of inclusive decay distributions, and the  $\bar{B} \rightarrow X_s \gamma$  photon spectrum are discussed.

## 1. Introduction

Testing the Cabibbo–Kobayashi–Maskawa (CKM) description of quark mixing and  $CP$  violation is a large part of the high energy experimental program in the near future. The goal is to overconstrain the unitarity triangle by directly measuring its sides and (some) angles in several decay modes. If the value of  $\sin 2\beta$ , the  $CP$  asymmetry in  $B \rightarrow \psi K_S$ , is near the CDF central value [1], then searching for new physics will require precise measurements of the magnitudes and phases of the CKM matrix elements. Inclusive  $B$  decay rates can give information on  $|V_{cb}|$ ,  $|V_{ub}|$ ,  $|V_{ts}|$ , and  $|V_{td}|$ .

The theoretical reliability of inclusive measurements can be competitive with the exclusive ones. For example, for the determination of  $|V_{cb}|$  model dependence enters at the same order of  $\Lambda_{\text{QCD}}^2/m_{c,b}^2$  corrections from both the inclusive semileptonic  $\bar{B} \rightarrow X_c e \bar{\nu}$  width and the  $\bar{B} \rightarrow D^* e \bar{\nu}$  rate near zero recoil. It is then important to test the theoretical ingredients of these analyses via other measurements.

The main uncertainty in theoretical predictions for inclusive  $B$  decay rates arise from the poorly known quark masses which define the phase space, and the bad behavior of the series of perturbative corrections when it is written in terms of the pole mass. Only the product of these quantities, the decay widths, are well-defined physical quantities; while perturbative multi-loop calculations are most conveniently done in terms of the pole mass. Of course, one would like to eliminate any quark mass from the predictions in favor of physical observables. Here we present a new method of eliminating  $m_b$  in terms of the  $\Upsilon(1S)$  meson mass [2].

## 2. Upsilon Expansion

Let us consider, for example, the inclusive  $\bar{B} \rightarrow X_u e \bar{\nu}$  decay rate. At the scale  $\mu = m_b$ ,

$$\Gamma(B \rightarrow X_u e \bar{\nu}) = \frac{G_F^2 |V_{ub}|^2}{192\pi^3} m_b^5 \left[ 1 - 2.41 \frac{\alpha_s}{\pi} \epsilon - 3.22 \frac{\alpha_s^2}{\pi^2} \beta_0 \epsilon^2 - \dots - \frac{9\lambda_2 - \lambda_1}{2m_b^2} + \dots \right]. \quad (1)$$

The variable  $\epsilon \equiv 1$  denotes the order in the modified expansion, and  $\beta_0 = 11 - 2n_f/3$ . In comparison, the expansion of the  $\Upsilon(1S)$  mass in terms of  $m_b$  has a different structure,

$$m_\Upsilon = 2m_b \left\{ 1 - \frac{(\alpha_s C_F)^2}{8} \left[ \epsilon + \frac{\alpha_s}{\pi} \left( \ell + \frac{11}{6} \right) \beta_0 \epsilon^2 + \dots \right] \right\}, \quad (2)$$

where  $\ell = \ln[\mu/(m_b \alpha_s C_F)]$  and  $C_F = 4/3$ . In this expansion we assigned to each term one less power of  $\epsilon$  than the power of  $\alpha_s$ . It is also convenient to choose the same renormalization scale  $\mu$ . The prescription of counting  $\alpha_s^n$  in  $B$  decay rates as order  $\epsilon^n$ , and  $\alpha_s^n$  in  $m_\Upsilon$  as order  $\epsilon^{n-1}$  is the Upsilon expansion. It combines different orders in the  $\alpha_s$  perturbation series in Eqs. (1) and (2), but as it is sketched below, this is the consistent way of combining these expressions.

At large orders in perturbation theory, the coefficient of  $\alpha_s^n$  in Eq. (1) has a part which grows as  $Cn! \beta_0^{n-1}$ . For large  $n$ , this divergence is cancelled by the  $\alpha_s^{n+1}$  term in Eq. (2), whose coefficient behaves as  $(1/\alpha_s)(C/5)n! \beta_0^{n-1}$  [3–5]. The crucial  $1/\alpha_s$  factor arises because the coefficient of  $\alpha_s^{n+1}$  in Eq. (2) contains a series of the form

$(\ell^{n-1} + \ell^{n-2} + \dots + 1)$  which exponentiates for large  $n$  to give  $\exp(\ell) = \mu/(m_b \alpha_s C_F)$ , and corrects the mismatch of the power of  $\alpha_s$  between the two series.

The infrared sensitivity of Feynman diagrams can be studied by introducing a fictitious infrared cutoff  $\lambda$ . The infrared sensitive terms are non-analytic in  $\lambda^2$ , such as  $(\lambda^2)^{n/2}$  or  $\lambda^{2n} \ln \lambda^2$ , and arise from the low momentum part of Feynman diagrams. Linear infrared sensitivity (terms of order  $\sqrt{\lambda^2}$ ) are a signal of  $\Lambda_{\text{QCD}}$  effects, quadratic sensitivity (terms of order  $\lambda^2 \ln \lambda^2$ ) are a signal of  $\Lambda_{\text{QCD}}^2$  effects, etc. From Refs. [5, 6] it follows that the linear infrared sensitivity cancels in the upilon expansion to order  $\epsilon^2$  (probably to all orders as well, but the demonstration of this appears highly non-trivial).

Thus, the upilon expansion is theoretically consistent both at large orders for the terms containing the highest possible power of  $\beta_0$ , and to order  $\epsilon^2$  including non-Abelian contributions.

The most important uncertainty in this approach is the size of nonperturbative contributions to  $m_\Upsilon$  other than those which can be absorbed into  $m_b$ . If the mass of heavy quarkonia can be computed in an operator product expansion then this correction is of order  $\Lambda_{\text{QCD}}^4/(\alpha_s m_b)^3$  by dimensional analysis. Quantitative estimates, however, vary in a large range, and it is preferable to constrain such effects from data. We use 100 MeV to indicate the corresponding uncertainty. Finally, if the nonperturbative contribution to  $\Upsilon$  mass,  $\Delta_\Upsilon$ , were known, it could be included by replacing  $m_\Upsilon$  by  $m_\Upsilon - \Delta_\Upsilon$  on the left hand side of Eq. (2).

There are three surprising facts that are either accidental or indicate that the nonperturbative contributions may be small: 1) applications in terms of the  $\Upsilon(2S)$  mass give consistent results [2]; 2) the  $D \rightarrow X e \nu$  rate in terms of the  $\psi$  mass works (un)reasonably well [2]; 3) sum rule calculations for  $e^+ e^- \rightarrow b \bar{b}$  find that the 1S  $b$  quark mass (defined as half of the perturbative part of  $m_{\Upsilon(1S)}$ ) is only 20 MeV different from  $m_{\Upsilon(1S)}/2$  [7].

### 3. Application

Substituting Eq. (2) into Eq. (1) and collecting terms of a given order in  $\epsilon$  gives

$$\Gamma(\bar{B} \rightarrow X_u e \bar{\nu}) = \frac{G_F^2 |V_{ub}|^2}{192\pi^3} \left( \frac{m_\Upsilon}{2} \right)^5 \times \left[ 1 - 0.115\epsilon - 0.031\epsilon^2 - \dots \right]. \quad (3)$$

The complete order  $\alpha_s^2$  result calculated recently [8] is included. Keeping only the part proportional to  $\beta_0$ , the coefficient of  $\epsilon^2$  would be  $-0.035$ . The

perturbation series,  $1 - 0.115\epsilon - 0.031\epsilon^2$ , is better behaved than the series in terms of the  $b$  quark pole mass,  $1 - 0.17\epsilon - 0.10\epsilon^2$ , or the series expressed in terms of the  $\overline{\text{MS}}$  mass,  $1 + 0.30\epsilon + 0.13\epsilon^2$ . The uncertainty in the decay rate using Eq. (3) is much smaller than that in Eq. (1), both because the perturbation series is better behaved, and because  $m_\Upsilon$  is better known (and better defined) than  $m_b$ . The relation between  $|V_{ub}|$  and the total semileptonic  $\bar{B} \rightarrow X_u e \bar{\nu}$  decay rate is [2]

$$|V_{ub}| = (3.04 \pm 0.06 \pm 0.08) \times 10^{-3} \times \left( \frac{\mathcal{B}(\bar{B} \rightarrow X_u e \bar{\nu}) 1.6 \text{ ps}}{0.001 \tau_B} \right)^{1/2}, \quad (4)$$

The first error is obtained by assigning an uncertainty in Eq. (3) equal to the value of the  $\epsilon^2$  term and the second is from assuming a 100 MeV uncertainty in Eq. (2). The scale dependence of  $|V_{ub}|$  due to varying  $\mu$  in the range  $m_b/2 < \mu < 2m_b$  is less than 1%. The uncertainty in  $\lambda_1$  makes a negligible contribution to the total error. Although  $\mathcal{B}(\bar{B} \rightarrow X_u e \bar{\nu})$  cannot be measured without significant experimental cuts, for example, on the hadronic invariant mass, this method will reduce the uncertainties in such analyses as well.

The  $\bar{B} \rightarrow X_c e \bar{\nu}$  decay depends on both  $m_b$  and  $m_c$ . It is convenient to express the decay rate in terms of  $m_\Upsilon$  and  $\lambda_1$  instead of  $m_b$  and  $m_c$ , using Eq. (2) and

$$m_b - m_c = \bar{m}_B - \bar{m}_D + \left( \frac{\lambda_1}{2\bar{m}_B} - \frac{\lambda_1}{2\bar{m}_D} \right) + \dots, \quad (5)$$

where  $\bar{m}_B = (3m_{B^*} + m_B)/4 = 5.313 \text{ GeV}$  and  $\bar{m}_D = (3m_{D^*} + m_D)/4 = 1.973 \text{ GeV}$ . We then find

$$\Gamma(\bar{B} \rightarrow X_c e \bar{\nu}) = \frac{G_F^2 |V_{cb}|^2}{192\pi^3} \left( \frac{m_\Upsilon}{2} \right)^5 0.533 \times \left[ 1 - 0.096\epsilon - 0.029_{\text{BLM}}\epsilon^2 \right], \quad (6)$$

where the phase space factor has also been expanded in  $\epsilon$ , and the BLM [9] subscript indicates that only the corrections proportional to  $\beta_0$  have been kept. For comparison, the perturbation series in this relation written in terms of the pole mass is  $1 - 0.12\epsilon - 0.07_{\text{BLM}}\epsilon^2$  [10]. Including the terms proportional to  $\lambda_{1,2}$ , Eq. (6) implies [2]

$$|V_{cb}| = (41.6 \pm 0.8 \pm 0.7 \pm 0.5) \times 10^{-3} \times \eta_{\text{QED}} \left( \frac{\mathcal{B}(\bar{B} \rightarrow X_c e \bar{\nu}) 1.6 \text{ ps}}{0.105 \tau_B} \right)^{1/2}, \quad (7)$$

where  $\eta_{\text{QED}} \sim 1.007$  is the electromagnetic radiative correction. The uncertainties come from

assuming an error in Eq. (6) equal to the  $\epsilon^2$  term, a  $0.25 \text{ GeV}^2$  error in  $\lambda_1$ , and a  $100 \text{ MeV}$  error in Eq. (2), respectively. The second uncertainty can be reduced by determining  $\lambda_1$  (see below).

The theoretical uncertainty hardest to quantify in the predictions for inclusive  $B$  decays is the size of quark-hadron duality violation. This was neglected in Eqs. (4) and (7). It is believed to be exponentially suppressed in the  $m_b \rightarrow \infty$  limit, but its size is poorly known for the physical  $b$  quark mass. Studying the shapes of inclusive decay distributions is the best way to constrain this experimentally, since duality violation would probably show up in a comparison of different spectra. The shape of the lepton energy [11–14] and hadron invariant mass [15, 16, 14] spectra in semileptonic  $\bar{B} \rightarrow X_c e \bar{\nu}$  decay, and the photon spectrum in  $\bar{B} \rightarrow X_s \gamma$  [17–21] can also be used to determine the heavy quark effective theory (HQET) parameters  $\bar{\Lambda}$  and  $\lambda_1$ . The extent to which these determinations agree with one another will indicate at what level to trust predictions for inclusive rates.

Last year CLEO measured the first two moments of the hadronic invariant mass-squared ( $s_H$ ) distribution,  $\langle s_H - \bar{m}_D^2 \rangle$  and  $\langle (s_H - \bar{m}_D^2)^2 \rangle$ , subject to the constraint  $E_e > 1.5 \text{ GeV}$  [22]. Here  $\bar{m}_D = (m_D + 3m_{D^*})/4$ . Each of these measurements give an allowed band on the  $\bar{\Lambda} - \lambda_1$  plane. Their intersection gives (at order  $\alpha_s$ ) [22]

$$\begin{aligned} \bar{\Lambda} &= (0.33 \pm 0.08) \text{ GeV}, \\ \lambda_1 &= -(0.13 \pm 0.06) \text{ GeV}^2. \end{aligned} \quad (8)$$

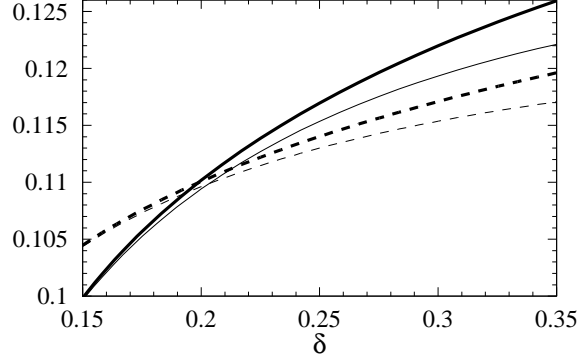
This agrees well with the analysis of the lepton energy spectrum in Ref. [12], although the order  $\Lambda_{\text{QCD}}^3/m_b^3$  corrections not included in these analyses introduce large additional uncertainties [14, 16].<sup>†</sup>

In the upilon expansion  $\bar{\Lambda}$  is not a free parameter, so we can determine  $\lambda_1$  directly with smaller uncertainty. Considering the observable [12]  $R_1 = \int_{1.5 \text{ GeV}} E_e (d\Gamma/dE_e) dE_e / \int_{1.5 \text{ GeV}} (d\Gamma/dE_e) dE_e$ , a fit to the same data yields [2]

$$\lambda_1 = (-0.27 \pm 0.10 \pm 0.04) \text{ GeV}^2. \quad (9)$$

This is in perfect agreement with the value of  $\lambda_1$  implied by the CLEO result for  $\langle s_H - \bar{m}_D^2 \rangle$  in the upilon expansion. The central value in Eq. (9) includes corrections of order  $\alpha_s^2/\beta_0$  [13], but the result at tree level or at order  $\alpha_s$  changes by less than  $0.03 \text{ GeV}^2$ . The first error is dominated by

<sup>†</sup> CLEO studied moments of the lepton spectrum [22], but the band corresponding to  $\langle E_e \rangle = (1.36 \pm 0.01 \pm 0.02) \text{ GeV}$  on the CLEO plot cannot be reproduced using the formulae in Ref. [11]. So I consider only the result in Eq. (8) from Ref. [22]. (I thank Iain Stewart for checking this calculation.)



**Figure 1.** Prediction for  $\overline{(1-x_B)}|_{x_B > 1-\delta}$  in the upilon expansion at order  $(\epsilon^2)_{\text{BLM}}$  (thick solid curve) and  $\epsilon$  (thick dashed curve). The thin curves show the contribution of the  $O_7$  operator only. (From Ref. [18].)

$1/m_b^3$  corrections [14] not included in Eq. (8). We varied the dimension-six matrix elements between  $\pm(0.5 \text{ GeV})^3$ , and combined their coefficients in quadrature. The second error is from assuming a  $100 \text{ MeV}$  uncertainty in Eq. (2).

Another way to test the upilon expansion, or determine the nonperturbative contribution to  $m_\gamma$ , is from  $\bar{B} \rightarrow X_s \gamma$ . Possible contributions to the total rate from physics beyond the standard model are unlikely to significantly affect the shape of the spectrum. The upilon expansion yields parameter free predictions for moments of this distribution. Experimentally one needs to make a lower cut on  $E_\gamma$ , so it is most convenient to study

$$\overline{(1-x_B)}|_{x_B > 1-\delta} = \frac{\int_{1-\delta}^1 dx_B (1-x_B) \frac{d\Gamma}{dx_B}}{\int_{1-\delta}^1 dx_B \frac{d\Gamma}{dx_B}}, \quad (10)$$

where  $x_B = 2E_\gamma/m_B$ . The parameter  $\delta = 1 - 2E_\gamma^{\text{min}}/m_B$  has to satisfy  $\delta > \Lambda_{\text{QCD}}/m_B$ , otherwise nonperturbative effects are not under control. Order  $\alpha_s^2/\beta_0$  corrections to the photon spectrum away from its endpoint were computed recently [18].

Fig. 1 shows the prediction for  $\overline{(1-x_B)}|_{x_B > 1-\delta}$  as a function of  $\delta$ , both at order  $\epsilon$  and  $(\epsilon^2)_{\text{BLM}}$ , neglecting nonperturbative contributions to  $m_\gamma$ . A  $+100 \text{ MeV}$  contribution would increase  $\overline{(1-x_B)}$  by 7%, so measuring  $\overline{(1-x_B)}$  with such accuracy will have important implications for the physics of quarkonia as well as for  $B$  physics.

For  $E_\gamma > 2.1 \text{ GeV}$  Fig. 1 gives  $\overline{(1-x_B)} = 0.111$ , whereas the central value from CLEO [23] is around 0.093. Interestingly, including the CLEO data point in the  $1.9 \text{ GeV} < E_\gamma < 2.1 \text{ GeV}$  bin, the experimental central value of  $\overline{(1-x_B)}$  over the

Decay widths	Expansions in terms of	
	$m_b^{\text{pole}}$ and $\alpha_s$	$m_\Upsilon$ and $\epsilon$
$B \rightarrow X_c e \bar{\nu}$	$1 - 0.12\epsilon - 0.07\epsilon^2$	$1 - 0.10\epsilon - 0.03\epsilon^2$
$B \rightarrow X_u e \bar{\nu}$	$1 - 0.17\epsilon - 0.13\epsilon^2$	$1 - 0.12\epsilon - 0.03\epsilon^2$
$B \rightarrow X_c \tau \bar{\nu}$	$1 - 0.10\epsilon - 0.06\epsilon^2$	$1 - 0.07\epsilon - 0.02\epsilon^2$
$B \rightarrow X_u \tau \bar{\nu}$	$1 - 0.16\epsilon$	$1 - 0.08\epsilon$
$B \rightarrow X_{c\bar{u}}(s+d)$	$1 - 0.05\epsilon - 0.04\epsilon^2$	$1 - 0.03\epsilon - 0.01\epsilon^2$
$B \rightarrow X_{c\bar{c}}(s+d)$	$1 + 0.20\epsilon + 0.15\epsilon^2$	$1 + 0.16\epsilon + 0.07\epsilon^2$
$B \rightarrow X_{u\bar{u}}(s+d)$	$1 - 0.10\epsilon$	$1 - 0.05\epsilon$
$B \rightarrow X_{u\bar{c}}(s+d)$	$1 + 0.09\epsilon$	$1 + 0.11\epsilon$

**Table 1.** Comparison of the perturbation series for inclusive decay rates using the conventional expansion and the upilon expansion [2]. The second order terms are the BLM parts only.

region  $E_\gamma > 1.9 \text{ GeV}$  is 0.117, whereas the upilon expansion predicts 0.120. Ultimately, one would like to see whether prediction and data agree over some range of the cut  $E_\gamma^{\text{min}}$ . One can also evaluate  $\overline{(1 - x_B)}$  in terms of  $\bar{\Lambda}$  and  $\lambda_1$  without using the upilon expansion. The CLEO data [23] in the region  $E_\gamma > 2.1 \text{ GeV}$  implies the central values  $\bar{\Lambda}_{\alpha_s} \simeq 390 \text{ MeV}$  and  $\bar{\Lambda}_{\alpha_s^2 \beta_0} \simeq 270 \text{ MeV}$  at order  $\alpha_s$  and  $\alpha_s^2 \beta_0$ , respectively [18]. The BLM terms may not dominate at order  $\alpha_s^2$  [21], so it is important to calculate the complete two-loop correction to the  $O_7$  contribution to  $\overline{(1 - x_B)}$ .

The upilon expansion has been applied to form factor ratios in exclusive semileptonic  $B$  decays, as well as nonleptonic decays [2], where it also improves the perturbation series (see Table 1). However, the semileptonic  $B$  branching fraction or the average number of charm quarks in  $B$  decay agree with other predictions in the literature. Applications of similar ideas for  $e^+e^- \rightarrow t\bar{t}$  are reviewed by Teubner at this Conference [24].

#### 4. Conclusions

- Using  $m_\Upsilon$  and the upilon expansion, i.e., assigning order  $\epsilon^n$  to the order  $\alpha_s^n$  term in  $B$  decay rates and  $\epsilon^{n-1}$  to the  $\alpha_s^n$  term in the perturbative expression for  $m_\Upsilon$ , is equivalent to using a short distance  $b$  quark mass.
- It improves the behavior of perturbation series for inclusive  $B$  decays, and eliminates  $m_b$  altogether from the theoretical predictions in favor of  $m_\Upsilon$  in a simple and consistent manner.
- It may lead to smaller nonperturbative effects (to the extent this is reflected in the behavior of perturbation series).

- The biggest uncertainty is the nonperturbative contribution to  $m_\Upsilon$  unrelated to the quark mass. It will be possible to estimate / constrain this from data in the future.

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